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Department of ECE

15UEC904
LINEAR CONTROL ENGINEERING

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State Variable Analysis

- State space representation of Continuous Time systems – State equations – Transfer function from State Variable Representation – Solutions of the state equations - Concepts of Controllability and Observability – State space representation for Discrete time system

UNIT - V STATE VARIABLE ANALYSIS

The state Variable approach is a powerful tool / technique for analysis ~~of~~ and design of control system.

The state space Analysis is a modern approach and also easier for analysis using digital computers.

→ conventional method is Transfer function approach

Drawback of TF. approach

1. TF is defined under zero initial condition
2. TF is applicable to linear time invariant system.
3. TF analysis is restricted to single input and single O/P system.
4. Does not provide in formation regarding the internal state of the system.

State : The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$ completely determines the behaviour of the system for $t \geq t_0$.

State variables:: The state variables of a dynamic system are the smallest set of variables that completely determines the state of the dynamic system.

State vector:: This is a vector consisting of n number of state variables that completely determine the behavior of a dynamic system.

state Space :- The state space is an n-dimensional space whose coordinate axes are the n-number of state variables that completely determine the behaviour of a dynamic system.

Any state can be represented by a point in the state space.

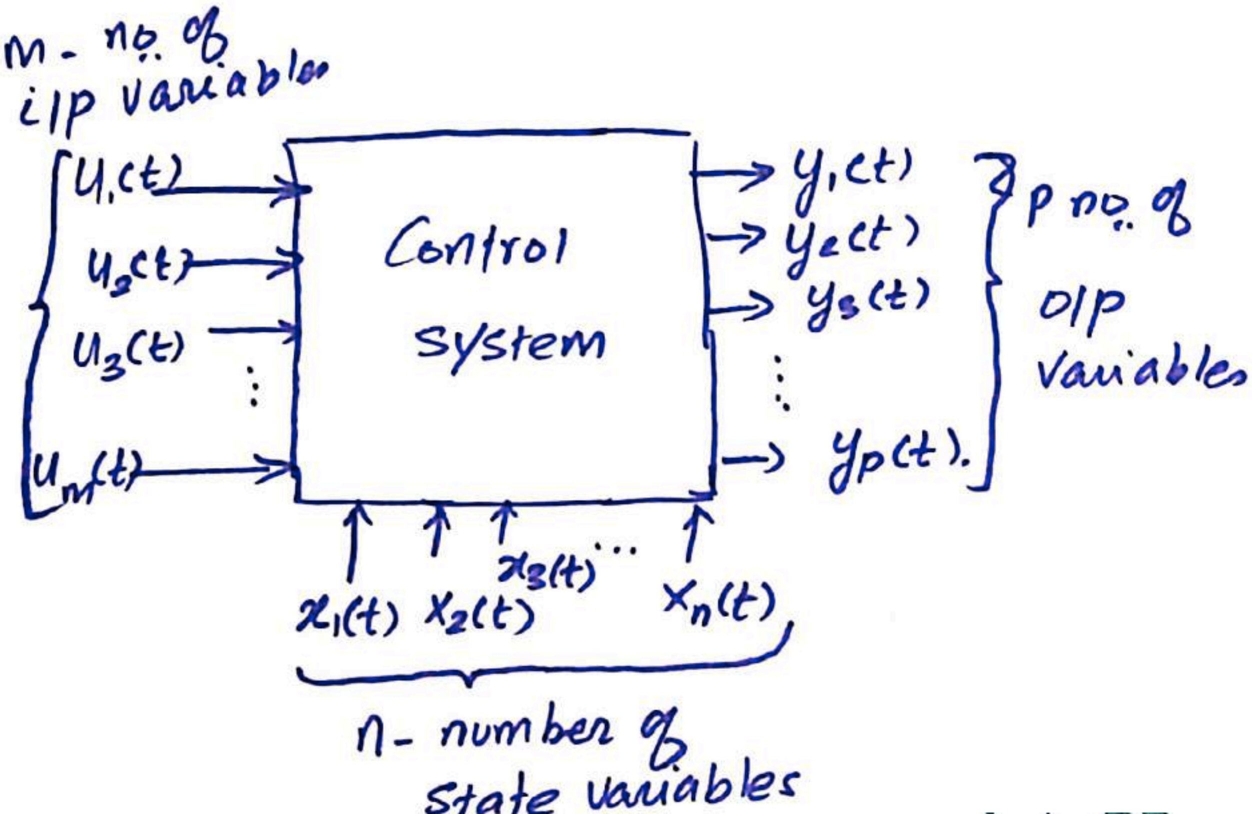
State space Equation: The input Variables the output variables and State variables are the three type of Variables used in state space modelling of dynamic System.

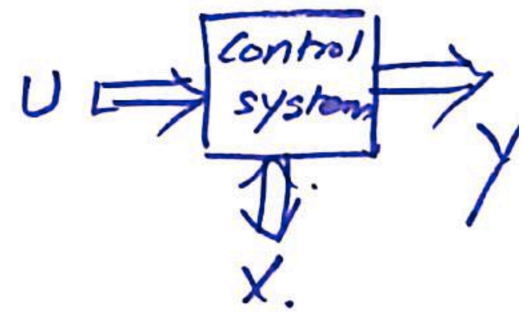
State space Formulation

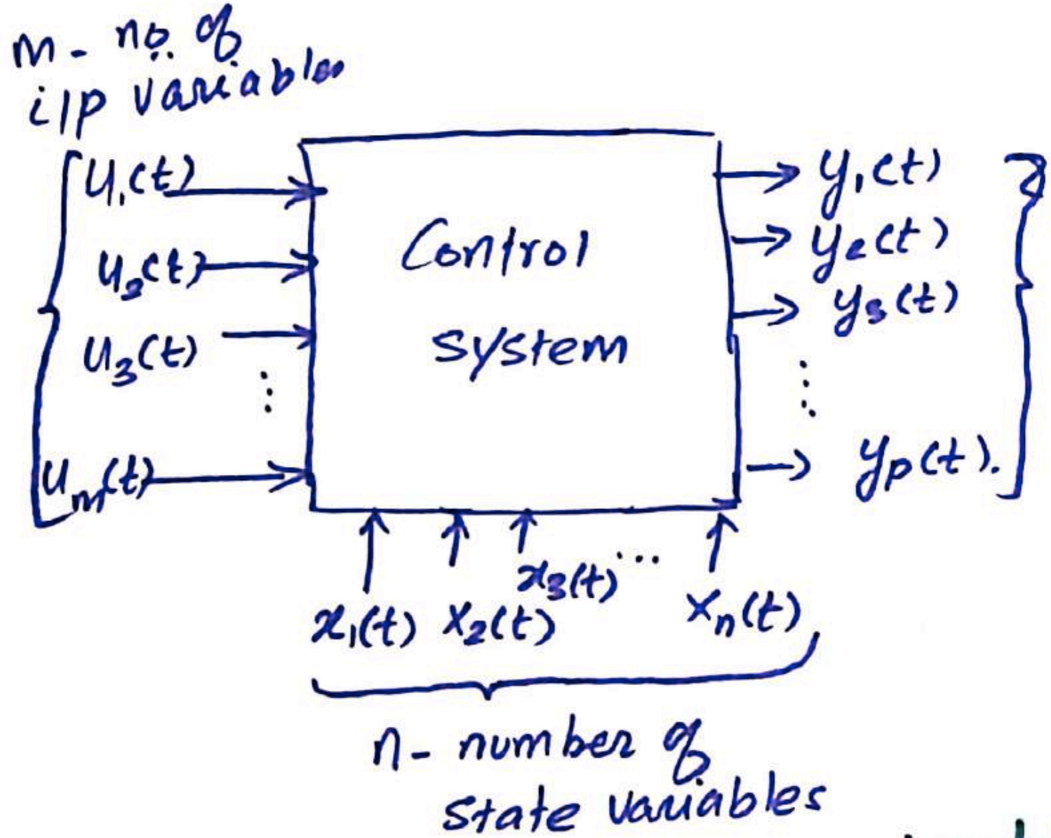
Let State variables = $x_1(t), x_2(t) \dots x_n(t)$

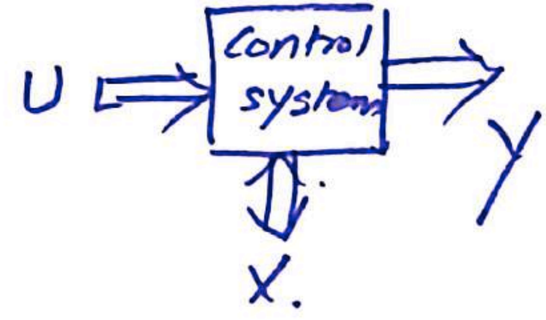
Input Variables : $u_1(t), u_2(t) \dots u_m(t)$

Output Variables : $y_1(t), y_2(t) \dots y_p(t)$.









Input vector

$$U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

output vector

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

State variable
Vector

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

State Equations

The state variable representation canbe arranged in the form of n- number of first order differential equation as shown below.

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2 \dots x_n ; u_1, u_2 \cdots u_m).$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2 \dots x_n ; u_1, u_2 \cdots u_m).$$

$$\vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2 \cdots x_n ; u_1, u_2 \cdots u_m).$$

The n-number of differential equations may be written in vector notation as

$$\dot{X}(t) = f(X(t), U(t)).$$

└ i/p space at time t.

└ state space of the system at t.

$Y(t) \rightarrow$ o/p ~~of~~ space of the system for the input space $X(t)$ at time t.

State model of Linear system.

Consist of state equation and o/p equation.

For LTI system, the 1st order derivatives of state variable can be expressed as linear combination of state variables & i/ps.

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots b_{2m}u_m$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots b_{nm}u_m$$

where the coefficients a_{ij} and b_{ij} are constant.

In matrix form

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
a_{31} & a_{32} & \cdots & a_{3n} \\
\vdots & & & \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}
+
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1m} \\
b_{21} & b_{22} & \cdots & b_{2m} \\
b_{31} & b_{32} & \cdots & b_{3m} \\
\vdots & & & \\
b_{n1} & b_{n2} & \cdots & b_{nm}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix}
$$

The matrix Equation $\dot{X}(t) = AX(t) + BU(t)$.

state Equation of LTI

$X(t) \rightarrow$ state vector of order $(n \times 1)$

$U(t) \rightarrow$ Input vector of order $(m \times 1)$

$A \rightarrow$ system matrix of order $(n \times n)$

$B \rightarrow$ Input matrix of order $(n \times m)$.

The o/p ~~vector~~ at any time are function of state variables and inputs.

O/p vector $Y(t) = f(X(t), U(t))$

o/p variables canbe expressed as a linear combination of state variables and inputs.

$$y_1 = c_{11}x_1 + c_{12}x_2 + \dots c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \dots + d_{1m}u_m$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + \dots c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \dots d_{2m}u_m$$

$$\vdots$$

$$y_p = c_{p1}x_1 + c_{p2}x_2 + \dots c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \dots d_{pm}u_m$$

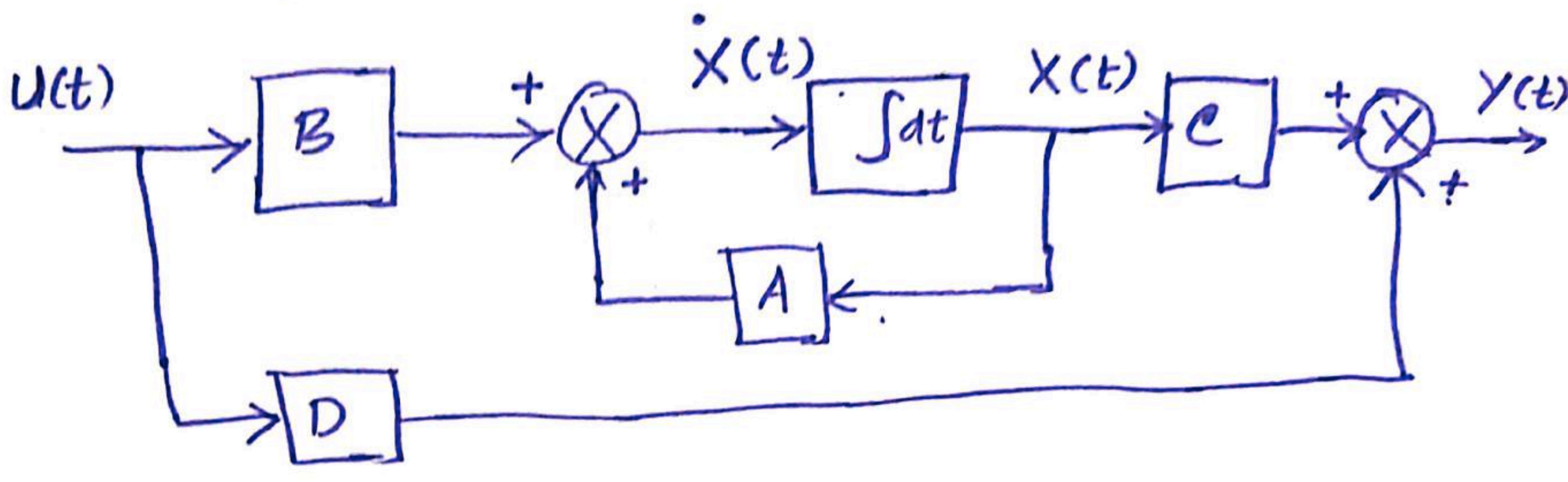
Here c_{ij} & d_{ij} are const. coeff.

State Model of LTI. system is given by

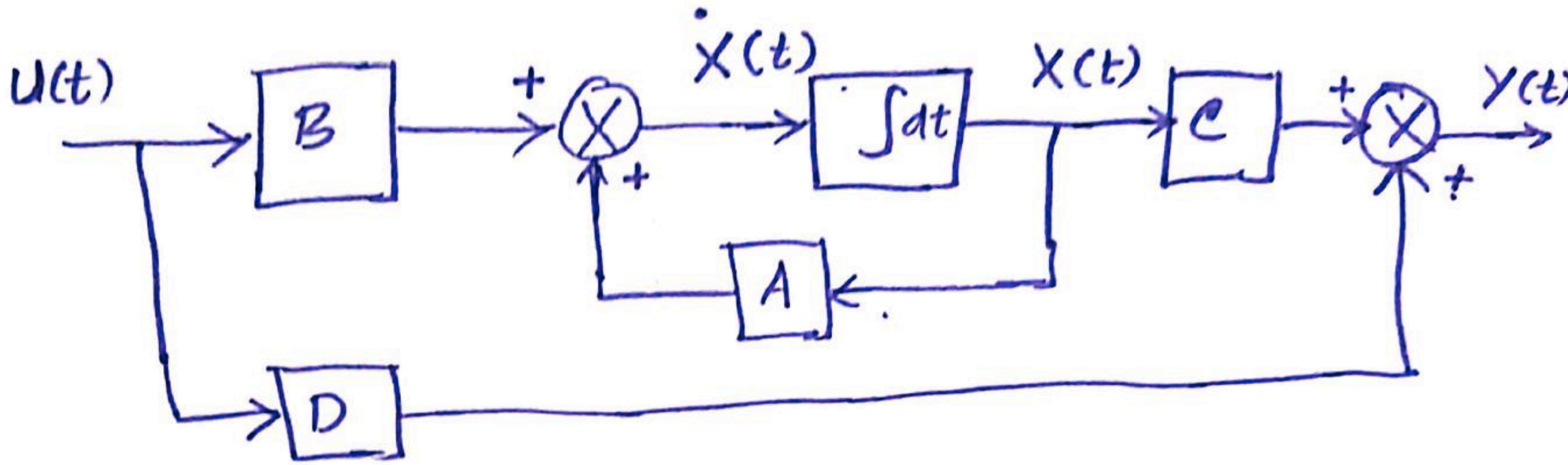
$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

$$Y(t) = C\,X(t) + D\,U(t).$$

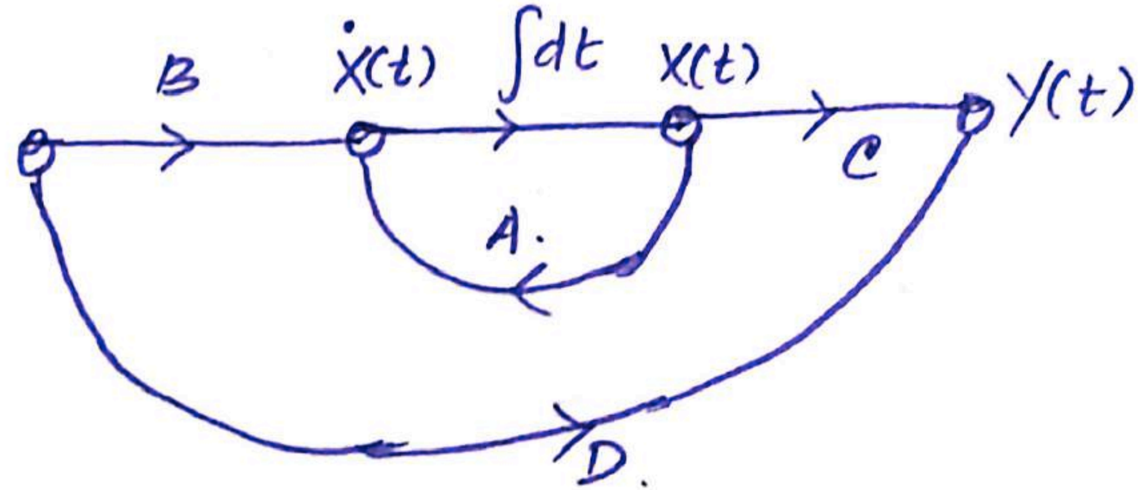
Block diagram of the State Equation is given below



Block diagram of the State Equation is given below

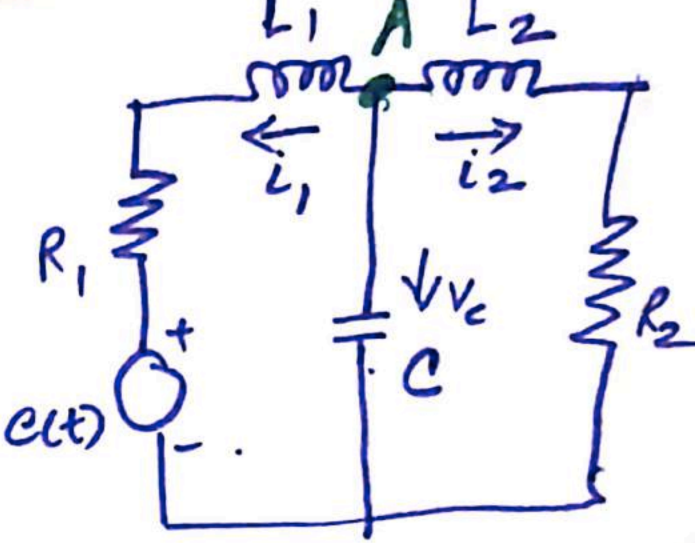


Signal Flow graph.



Example :-

obtain the state model of the electrical network shown. (minimal no. of state variable).



Sol^n

choose current through inductance i_1 & i_2,
Voltage across the capacitance V_c as
state variables

Let $x_1 = i_1 \rightarrow$ current through L_1

$x_2 = i_2 \rightarrow$ " " " L_2

$x_3 = V_c \rightarrow$ Voltage across capacitor

Apply Kirchoff current law at Node A.

$$i_1 + i_2 + C\frac{dV_c}{dt} = 0$$

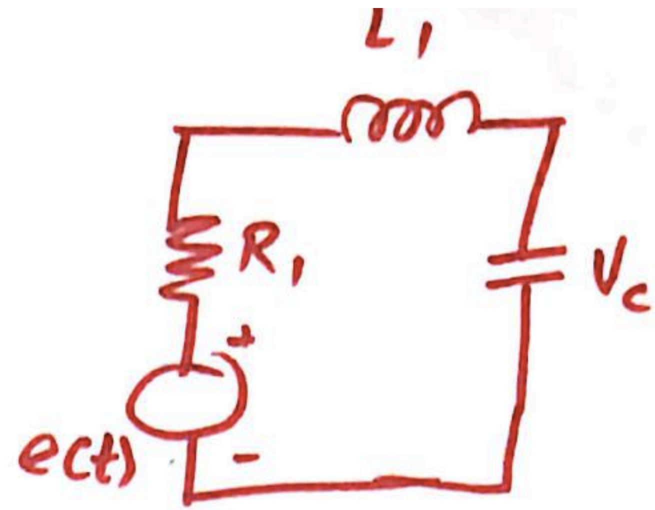
Subitute state variables

$$x_1 + x_2 + C\dot{x}_3 = 0.$$

$$C\dot{x}_3 = -x_1 - x_2$$

$$\Rightarrow \boxed{\dot{x}_3 = -\frac{1}{C}x_1 - \frac{1}{C}x_2}$$

By Kirchoff Voltage law



$$e(t) + i_1 R_1 + L_1 \frac{di_1}{dt} = V_c.$$

sub. state Variable

$$e(t) + x_1 R_1 + L_1 \dot{x}_1 = X_3$$

Let $e(t) = u(t) \rightarrow u.$

$$\Rightarrow \quad u + x_1 R_1 + L_1 \dot{x}_1 = X_3$$

$$L_1 \dot{x}_1 = -x_1 R_1 + X_3 - u.$$

$$\dot{x}_1 = -\frac{R_1}{L_1} x_1 + \frac{1}{L_1} X_3 - \frac{1}{L_1} u \quad \text{— ② state Eq.}$$

By kirchoff voltage law

$$V_c = L_2 \frac{di_2}{dt} + i_2 R_2$$

Apply state variable

$$X_3 = L_2 \dot{x}_2 + x_2 R_2$$

$$L_2 \dot{x}_2 = X_3 - X_2 R_2$$

$$\Rightarrow \dot{x}_2 = -\frac{R_2}{L_2} x_2 - \frac{1}{L_2} X_3 \quad (3)$$

State Equations are

$$\dot{x}_1 = -\frac{R_1}{L_1}x_1 + \frac{1}{L_1}x_3 - \frac{1}{L_1}u$$

$$\dot{x}_2 = -\frac{R_2}{L_2}x_2 + \frac{1}{L_2}x_3$$

$$\dot{x}_3 = -\frac{1}{C}x_1 - \frac{1}{C}x_2$$

Matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-\frac{R_1}{L_1} & 0 & \frac{1}{L_1}\\ 0 & \frac{-R_2}{L_2} & \frac{1}{L_2}\\ \frac{-1}{C} & -\frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}-\frac{1}{L_1}\\ 0\\ 0\end{bmatrix}\begin{bmatrix}u\end{bmatrix}$$

O/p Variables

Voltage across the ressistor as o/p variables denoted by y_1 & y_2

(u) $y_1 = i_1 R_1$; $y_2 = i_2 R_2$

State $y_1 = x_1 R_1$; $y_2 = x_2 R_2$

Matrix.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & \frac{-R_2}{L_2} & \frac{1}{L_2} \\ -\frac{1}{C_1} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

Matrix.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example

Construct a state model for a system characterized by the differential Equation

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0.$$

Give Block diagram representation.

Solut

The state variables ~~ap~~ are x_1, x_2 & x_3 (phase variables)

Let $x_1 = y$

$$x_2 = \frac{dy}{dt}$$

$$x_3 = \frac{d^2y}{dt^2} = \dot{x}_2$$

$$\dot{x}_3 = \frac{d^3y}{dt^2} = \ddot{x}_2$$

Apply the State ~~equation~~ variables in the System Equation.

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$$

$$\dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0.$$

$$\Rightarrow \quad \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0.$$

or, $$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u.$$

State eqn

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u.$$

State equ

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u.$

By writing into matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u].$$

Here y is O/P.

$y = x_1.$

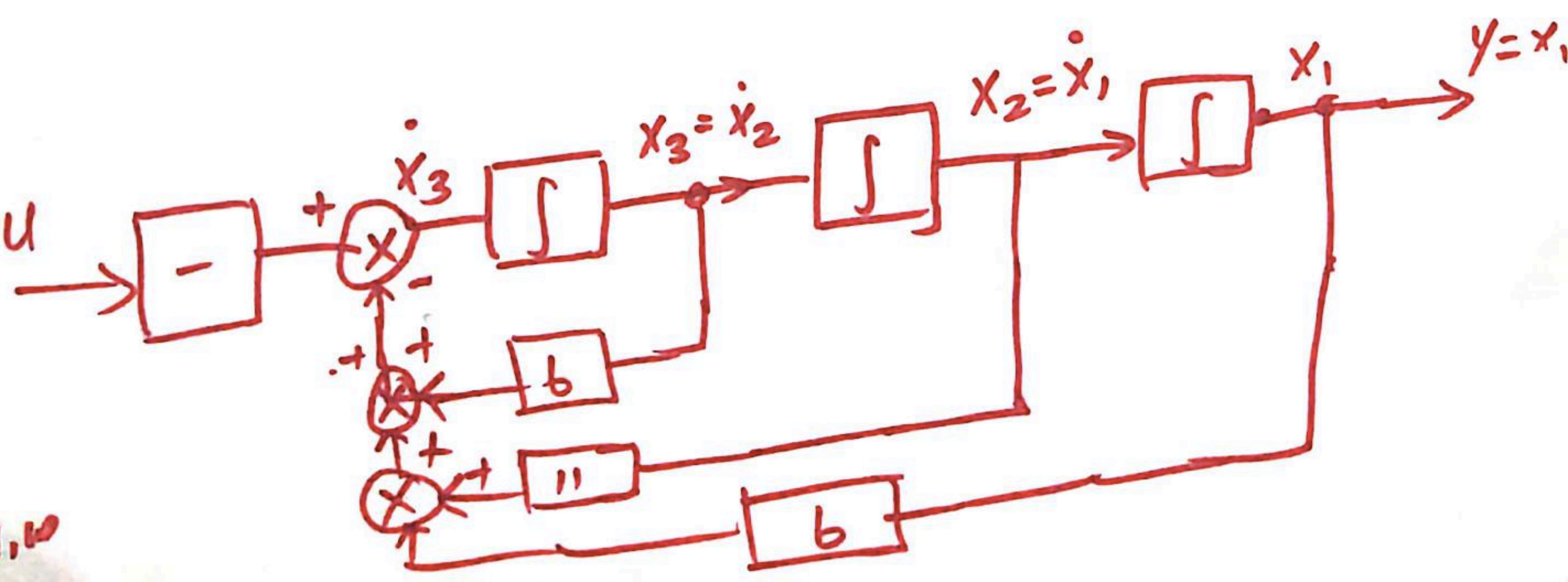
The O/P equation is $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

State eqn

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u.$$



Example

Obtain the state model of the system whose transfer function is given as

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}.$$

Sol.

$$Y(s)\left[s^3+4s^2+2s+1\right] = 10\,U(s)$$

$$s^3Y(s)+4s^2Y(s)+2sY(s)+Y(s) = 10\,U(s)$$

L.T

Solⁿ

$$Y(s)\left[s^3+4s^2+2s+1\right]=10\,U(s)$$

$$s^3Y(s)+4s^2Y(s)+2sY(s)+Y(s)=10\,U(s)$$

By taking inverse L.T.

$$\dddot{y}+4\ddot{y}+2\dot{y}+y=10u. \quad \text{— (1)}$$

Let define state variable:

$$x_1=y\;;\quad x_2=\dot{y}\;;\quad x_3=\ddot{y}.$$

$$\dot{x}_3=\dddot{y} \quad , \quad \text{— (1)}.$$

Let define state variable ; $x_3 = \ddot{y}$.

$x_1 = y$; $x_2 = \dot{y}$;

$\dot{x}_3 = \dddot{y}$ — ①.

Sub in the above equ

$\dot{x}_3 + 4x_3 + 2x_2 + x_1 = 10u$

$\dot{x}_3 = -4x_3 - 2x_2 - x_1 + 10u$

$\dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u$.

state equation

$$\dot{x}_1 = x_2 \, ;$$

$$\dot{x}_2 = x_3 \, ;$$

$$\dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u$$

O/p equation $y = x_1$

stat model in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} [u].$$

Thank You